THE KP HIERARCHY, BRANCHED COVERS, AND TRIANGULATIONS

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ABSTRACT. The KP hierarchy is a completely integrable system of quadratic, partial differential equations that generalizes the KdV hierarchy. A linear combination of Schur functions is a solution to the KP hierarchy if and only if its coefficients satisfy the Plücker relations from geometry. We give a solution to the Plücker relations involving products of variables marking contents for a partition, and thus give a new proof of a content product solution to the KP hierarchy, previously given by Orlov and Shcherbin. In our main result, we specialize this content product solution to prove that the generating series for a general class of transitive ordered factorizations in the symmetric group satisfies the KP hierarchy. These factorizations appear in geometry as encodings of branched covers, and thus by specializing our transitive factorization result, we are able to prove that the generating series for two classes of branched covers satisfies the KP hierarchy. For the first of these, the double Hurwitz series, this result has been previously given by Okounkov. The second of these, that we call the m-hypermap series, contains the double Hurwitz series polynomially, as the leading coefficient in m. The m-hypermap series also specializes further, first to the series for hypermaps and then to the series for maps, both in an orientable surface. For the latter series, we apply one of the KP equations to obtain a new and remarkably simple recurrence for triangulations in a surface of given genus, with a given number of faces. This recurrence leads to explicit asymptotics for the number of triangulations with given genus and number of faces, in recent work by Bender, Gao and Richmond.

1. INTRODUCTION AND BACKGROUND

1.1. **The KP hierarchy.** The KP (Kadomtsev-Petriashvili) hierarchy is a completely integrable system of quadratic, partial differential equations for an unknown function F, that generalizes the KdV hierarchy. Over the last two decades, there has been strong interest in the relationship between integrable systems and moduli spaces of curves. This began with Witten's Conjecture [W] for the KdV equations, proved by Kontsevich [Ko] (and more recently by a number of others, including [KL]). Pandharipande [P] conjectured that solutions to the Toda equations arose in a related context, which was proved by Okounkov [Ok], who also proved the more general result that a generating series for what he called *double Hurwitz numbers* satisfies the KP-hierarchy. Kazarian and Lando [KL], in their recent proof of Witten's conjecture, showed that it is implied by Okounkov's result for double Hurwitz numbers. More recently, Kazarian [Ka] has given a number of interesting results about the structure of solutions to the KP hierarchy.

To fix ideas, the first few of the countable equations in the KP hierarchy are

(1)
$$F_{2,2} - F_{3,1} + \frac{1}{12}F_{1^4} + \frac{1}{2}F_{1^2}^2 = 0,$$

$$F_{3,2} - F_{4,1} + \frac{1}{6}F_{2,1^3} + F_{1,1}F_{2,1} = 0,$$

$$F_{4,2} - F_{5,1} + \frac{1}{4}F_{3,1^3} - \frac{1}{120}F_{1^6} + F_{1^2}F_{3,1} + \frac{1}{2}F_{2,1}^2 - \frac{1}{8}F_{1^3}^2 - \frac{1}{12}F_{1^2}F_{1^4} = 0,$$

where

$$F_{r^{a_r},\ldots,1^{a_1}} := \frac{\partial^{a_r + \cdots + a_1}}{\partial p_r^{a_r} \cdots \partial p_1^{a_1}} F.$$

We have found [MJD] to be an accessible source for the KP hierarchy, and note that the variables p_i that we use in this paper are related to the variables used in [MJD] by $p_i = i\,x_i$, $i \geq 1$. Thus the partial derivatives in our equations (with respect the p_i's) are scaled differently from those in [MJD] (which are with respect to the x_i's).

There is a powerful characterization of the solutions to the KP hierarchy that is particularly convenient from an algebraic combinatorics or geometric point of view. This is well-known in the integrable systems literature (see, *e.g.*, Ch. 10 of [MJD]), and concerns linear combinations of Schur symmetric functions (here the variables $p_1, p_2, \ldots$ are the power sum symmetric functions). The characterization states that the coefficients in the linear combination of Schur functions satisfy the Plücker relations from geometry. In the remainder of Section 1 (Sections 1.2 – 1.4), we give the background required to state this characterization precisely (Theorem 1.3).

In Section 2, we apply the Schur function characterization to give an explicit solution to the KP hierarchy (as Theorem 2.3) in terms of products of a countable set of variables indexed by the integers, recording *content*. Content is a combinatorial parameter for partitions that often appears in expressions for symmetric functions (which are themselves naturally indexed by partitions). We establish this result as an immediate corollary of Theorem 2.2, in which we give a content product solution to the Plücker relations. The content product solution to the KP hierarchy has previously been given by Orlov and Shcherbin [OS], using different methods. In Section 3 we turn to algebraic combinatorics. We consider a general set of transitive ordered factorizations in the symmetric group, and prove that a particular generating function for the numbers of such factorizations satisfies the KP hierarchy. This is our main result, given as Theorem 3.1. In Section 4, we consider the geometrical interpretation of transitive ordered factorizations, in terms of branched covers. Thus in Section 4.2, as a corollary of our main result, we obtain Okounkov's [Ok] result, that the generating series for double Hurwitz numbers satisfies the KP hierarchy (Theorem 4.1). In Section 4.3 we consider a new class of geometric numbers, called m-hypermap numbers, and prove that the generating series for these satisfies the KP hierarchy (Theorem 4.2). In Section 4.4, we prove that the m-hypermap numbers are polynomials in m, with the double Hurwitz numbers as the leading coefficient (Theorem 4.3), and speculate that the rich geometry for the latter might extend to the former. Then in Section 5, we specialize further to establish that generating series for the numbers of rooted hypermaps (Theorem 5.1) and rooted maps in an orientable surface (Theorem 5.2) are solutions to the KP hierarchy. In Section 5.3, we conclude by using one

of the KP equations to give a new and remarkably simple recurrence for triangulations in an orientable surface of specified genus, with a given number of faces (Theorem 5.4).

1.2. **The Plücker relations.** If $\lambda_1, \ldots, \lambda_n$ are integers with $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$ and $\lambda_1 + \cdots + \lambda_n = d$, then $\lambda = (\lambda_1, \ldots, \lambda_n)$ is said to be a *partition* of $|\lambda| := d$ (indicated by writing $\lambda \vdash d$) with $l(\lambda) := n$ *parts*. The empty list ε of integers is to be regarded as a partition of $d = 0$ with $n = 0$ parts, and let $\mathcal{P}$ denote the set of all partitions. If λ has f_j parts equal to j for $j = 1, \ldots, d$, then we also write $\lambda = (d^{f_d}, \ldots, 1^{f_1})$, where convenient. Also, $\mathrm{Aut}\, \lambda$ denotes the set of permutations of the n positions that fix λ; therefore $|\mathrm{Aut}\, \lambda| = \prod_{j \geq 1} f_j!$.

We consider a set $\{b_\lambda \in \mathbb{Q}[u_1, u_2, \ldots] : \lambda \in \mathcal{P}\}$, where $(u_1, u_2, \ldots)$ is a list of indeterminates independent of $p_1, p_2, \ldots$. It is convenient to adopt two conventions for evaluating b_λ in certain cases when λ is not a partition.

Convention 1: For $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_n \leq 0$, then

(2) $\qquad b_\lambda := 0, \qquad \text{if } \lambda_n < 0; \qquad\qquad b_\lambda := a_{(\lambda_1 \ldots, \lambda_{n-1})}, \qquad \text{if } \lambda_n = 0.$

The second convention concerns arbitrary lists $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers (*i.e.*, not necessarily in weakly decreasing order). For such a list, we define the operator Δ_j, for each $j = 1, \ldots, n - 1$, by

(3) $\qquad \Delta_j\, \lambda = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1} - 1, \lambda_j + 1, \lambda_{j+2}, \ldots, \lambda_n).$

Convention 2:

(4) $\qquad\qquad\qquad\qquad b_\lambda := -b_{\Delta_j\, \lambda}, \qquad j = 1, \ldots, n - 1.$

Note that if $\Delta_j\, \lambda = \lambda$, which is equivalent to $\lambda_{j+1} = \lambda_j + 1$, then (4) implies that

(5) $\qquad\qquad\qquad\qquad b_\lambda = -b_\lambda = 0.$

It is now straightforward to determine the value of b_λ for any list $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers: if λ is not in weakly decreasing order, then we apply (4) until we either can apply (5), or until we have a list that is weakly decreasing; then we finish with (2) to remove any terminal 0's.

Let $\alpha, \beta \in \mathcal{P}$, where α and β have i and j parts, respectively, for some $i, j \geq 0$, $(i, j) \neq (0, 0)$, and suppose $\alpha = (\alpha_1, \ldots, \alpha_i)$ and $\beta = (\beta_1, \ldots, \beta_j)$. Let $m = \max\{i + 1, j - 1, 2\}$, and set $\alpha_{i+1} = \cdots = \alpha_{m-1} = \beta_{j+1} = \cdots = \beta_{m+1} = 0$. Then we say that $\{b_\lambda \in \mathbb{Q}[u_1, u_2, \ldots] : \lambda \in \mathcal{P}\}$ satisfies the *Plücker relations* if it satisfies the equation

(6) $\qquad \displaystyle\sum_{k=0}^{m} (-1)^k b_{(\alpha_1 - 1, \ldots, \alpha_{m-1} - 1, \beta_{k+1} + m - k)} \cdot b_{(\beta_1 + 1, \ldots, \beta_k + 1, \beta_{k+2}, \ldots, \beta_{m+1})} = 0,$

for each such pair of partitions α, β, subject to (2) and (4). Note that partitions are represented by *Maya diagrams* in [MJD], and the statement of the Plücker relations given above is a translation of the Maya diagram notation used in [MJD].

1.3. **Schur functions and characters of the symmetric group.** We recall a number of basic facts about symmetric functions (see [S]), where $x_1, x_2, \ldots$ are the underlying indeterminates. The ith *power sum* symmetric function is $p_i = \sum_{j \geq 1} x_j^i$, for $i \geq 1$, with $p_0 := 1$. The

ith *complete* symmetric function h_i is defined by $\sum_{i \geq 0} h_i t^i = \prod_{j \geq 1} (1 - x_j t)^{-1}$ and is related to the power sums through

(7)
$$\sum_{i \geq 0} h_i t^i = \exp \sum_{k \geq 1} \frac{p_k}{k} t^k.$$

The Schur function s_λ may be expressed in terms of the complete symmetric functions through the *Jacobi-Trudi formula*

(8)
$$s_\lambda = \det (h_{\lambda_i - i + j})_{i,j=1,\ldots,n} \, ,$$

(with the convention that $h_i = 0$ for $i < 0$). We write $h_i(p_1, p_2, \ldots)$ to denote the ith complete symmetric function of indeterminates for which the power sums are given by $p_1, p_2, \ldots$, and similarly for $s_\lambda(p_1, p_2, \ldots)$.

Let χ_μ^λ denote the *character* of the irreducible representation of $\mathfrak{S}_d$ indexed by λ, evaluated at any element of the conjugacy class $\mathcal{C}_\mu$ (we usually refer to χ_μ^λ informally as an irreducible character). Then, for $\lambda \vdash d$, the explicit expressions expressing the Shur functions and the power sums in terms of each other are

(9)
$$p_\lambda = \sum_{\mu \vdash d} \chi_\lambda^\mu s_\mu, \qquad s_\lambda = \sum_{\mu \vdash d} \frac{|\mathcal{C}_\mu|}{d!} \chi_\mu^\lambda p_\mu.$$

It is convenient to consider a particular scaling of the irreducible characters, given by

(10)
$$g_\mu^\lambda = |\mathcal{C}_\mu| \frac{\chi_\mu^\lambda}{\chi_{(1^d)}^\lambda}, \qquad \lambda, \mu \vdash d.$$

The following enumerative result is well-known, and is included here since it will be applied later.

Proposition 1.1. *For $\mu_i \vdash d \geq 1$ and $\sigma_i \in \mathcal{C}_{\mu_i}$, $i = 1, \ldots, k$, the number of k-tuples $(\sigma_1, \ldots, \sigma_k)$, that satisfy the equation $\sigma_1 \cdots \sigma_k = \iota$ (where ι is the identity permutation) is*

$$\frac{1}{d!} \sum_{\lambda \vdash d} \left(\chi_{(1^d)}^\lambda \right)^2 g_{\mu_1}^\lambda \cdots g_{\mu_k}^\lambda.$$

1.4. **A characterization of solutions to the KP hierarchy.** For any solution F to the KP hierarchy, the series e^F is called a τ-*function* of the KP hierarchy. The following result gives a characterization of τ-functions that is well-known in the integrable systems literature (see, *e.g.*, Ch. 10 of [MJD]).

Theorem 1.2. *The series*

$$\sum_\lambda b_\lambda s_\lambda(p_1, p_2, \ldots)$$

is a τ-function for the KP hierarchy if and only if $\{b_\lambda \in \mathbb{Q}[u_1, u_2, \ldots] : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations (6).

An obviously equivalent statement is the following.

Theorem 1.3. *The series*

$$\log \sum_\lambda b_\lambda s_\lambda(p_1, p_2, \ldots)$$

is a solution to the KP hierarchy if and only if $\{b_\lambda \in \mathbb{Q}[u_1, u_2, \ldots] : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations (6).

The coefficients b_λ in Theorems 1.2 and 1.3 are called the *Plücker coordinates* of the corresponding solution to the KP hierarchy.

2. A CONTENT PRODUCT SOLUTION TO THE KP HIERARCHY

2.1. **Content products for partitions.** Some preliminary results about Ferrers diagrams of partitions are required. The *Ferrers graph* for a partition λ is an array of unit boxes, called its *cells*, with the ith row from the top containing λ_i boxes, in columns (indexed from the left) $1, \ldots, \lambda_i$, for $i = 1, \ldots, n$. The *content* $c(w)$ of the cell w in row i and column j is $c(w) := j - i$. For indeterminates y_i, where i is an arbitrary integer, the *content product* for λ is

$$\text{(11)} \qquad C(\lambda) := \prod_{w \in \lambda} y_{c(w)},$$

where the the product is over all cells w of the Ferrers graph of λ. For example, $C(5, 3, 3, 2) = y_{-3}y_{-2}^2 y_{-1}^2 y_0^3 y_1^2 y_2 y_3 y_4$. It is to be remembered throughout that the indeterminates y_i may have negative suffices. Perhaps the best known formula involving contents is

$$\text{(12)} \qquad s_\lambda\big|_{p_i = x, i \geq 1} = \frac{d!}{\chi_{(1^d)}^\lambda} \prod_{w \in \lambda}(x + c(w)),$$

known as the *principal specialization* of the Schur function (see, *e.g.*, [S]), which has been recorded here since it will be applied later.

In addition, it will be convenient to consider other products of the y_i's. For pairs of integers m, k, we define $Y(m, k)$ by

$$\text{(13)} \qquad Y(m, k) := \begin{cases} \prod_{j=1}^k y_{m+1-j}, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0, \\ Y(m - k, -k)^{-1}, & \text{if } k \leq -1. \end{cases}$$

Clearly, for all integers m, j, k we have

$$\text{(14)} \qquad \frac{Y(m, k)}{Y(m, j)} = Y(m - j, k - j) = \frac{1}{Y(m - k, j - k)}.$$

Proposition 2.1. *Let*

$$\text{(15)} \qquad \mathcal{Y}(\lambda) := \prod_{i=1}^n Y(\lambda_i - i + 1, \lambda_i).$$

where λ is a list of integers of length $n \geq 2$. Then, for Δ_j defined in (3), we have

$$\mathcal{Y}(\Delta_j \lambda) = \mathcal{Y}(\lambda), \qquad j = 1, \ldots, n - 1.$$

Proof. From (15), (3) and (14), we have

$$
\begin{aligned}
\frac{\mathcal{Y}(\Delta_j \lambda)}{\mathcal{Y}(\lambda)} &= \frac{Y(\lambda_{j+1} - j, \lambda_{j+1} - 1)Y(\lambda_j - j + 1, \lambda_j + 1)}{Y(\lambda_j - j + 1, \lambda_j)Y(\lambda_{j+1} - j, \lambda_{j+1})} \\
&= \frac{Y(\lambda_j - j + 1, \lambda_j + 1)}{Y(\lambda_j - j + 1, \lambda_j)} \frac{Y(\lambda_{j+1} - j, \lambda_{j+1} - 1)}{Y(\lambda_{j+1} - j, \lambda_{j+1})} \\
&= Y(-j + 1, 1)\frac{1}{Y(-j + 1, 1)} = 1,
\end{aligned}
$$

giving the result.   □

Note that if λ is a partition then

(16) $$\mathcal{Y}(\lambda) = C(\lambda).$$

2.2. **A content product solution to the Plücker relations.** The following result gives an explicit class of solutions to the Plücker relations, involving the content product $C(\lambda)$ defined in (11). We have been unable to find this result stated explicitly in the literature.

Theorem 2.2.

$$
\{s_\lambda(q_1, q_2, \ldots) \prod_{w \in \lambda} y_{c(w)} \; : \; \lambda \in \mathcal{P}\}
$$

satisfies the Plücker relations.

Proof. For an arbitrary list λ of integers, we define f_λ by $f_\lambda := \mathcal{Y}(\lambda) s_\lambda(q_1, q_2, \ldots)$, through (8) and (15). We prove that $\{f_\lambda : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations (6). Here the indeterminates for the f_λ's are y_i for integers i, together with the q_j's for positive integers j. Note that the f_λ's satisfies conventions (2), immediately from (8), and (4) immediately from Proposition 2.1 and (8).

Local to this proof, we introduce the notation

$$\gamma^{(k)} := (\alpha_1 - 1, \ldots, \alpha_{m-1} - 1, \beta_{k+1} + m - k), \qquad \delta^{(k)} := (\beta_1 + 1, \ldots, \beta_k + 1, \beta_{k+2}, \ldots, \beta_{m+1}),$$

for $k = 0, \ldots, m$, and

$$\alpha' := (\alpha_1, \ldots, \alpha_{m-1}), \qquad \text{and} \qquad \beta' := (\beta_1, \ldots, \beta_{m+1}).$$

Then, for $k = 0, \ldots, m$, we obtain, applying (15) and (14),

$$
\begin{aligned}
\mathcal{Y}(\gamma_k)\mathcal{Y}(\delta_k) &= \mathcal{Y}(\alpha' - 1)\mathcal{Y}(\beta' + 1)\frac{\mathcal{Y}(\beta_{k+1} - k + 1, \beta_{k+1} + m - k)}{\mathcal{Y}(\beta_{k+1} - k + 1, \beta_{k+1} + 1)} \prod_{i=k+2}^{m+1} \frac{\mathcal{Y}(\beta_i - i + 2, \beta_i)}{\mathcal{Y}(\beta_i - i + 2, \beta_i + 1)} \\
&= \mathcal{Y}(\alpha' - 1)\mathcal{Y}(\beta' + 1)\frac{\mathcal{Y}(-k, m - k + 1)}{\prod_{i=k+2}^{m+1} \mathcal{Y}(-i + 2, 1)} \\
&= \mathcal{Y}(\alpha' - 1)\mathcal{Y}(\beta' + 1)y_{1-m}^{-1},
\end{aligned}
$$

where $\alpha' - 1$ is the list obtained from α' by subtracting 1 from every entry, and $\beta' + 1$ is the list obtained from β' by adding 1 to every entry. In particular, we have proved that $\mathcal{Y}(\gamma_k)\mathcal{Y}(\delta_k)$ is independent of k, so in checking the Plücker relations (6), we have

(17) $$\sum_{k=0}^{m}(-1)^k f_{\gamma^{(k)}} \cdot f_{\delta^{(k)}} = \mathcal{Y}(\alpha' - 1)\mathcal{Y}(\beta' + 1)y_{1-m}^{-1}\sum_{k=0}^{m}(-1)^k s_{\gamma^{(k)}} \cdot s_{\delta^{(k)}}.$$

Now, for $m \geq 1$ consider the matrices

$$\mathbf{A} = (h_{\alpha_i - i + j - 1})_{\substack{i=1,\ldots,m-1 \\ j=1,\ldots,m}}, \qquad \text{and} \qquad \mathbf{B} = (h_{\beta_i - i + j + 1})_{\substack{i=1,\ldots,m+1 \\ j=1,\ldots,m}},$$

where the h_k are in the power sums $q_1, q_2, \ldots$. Let $\mathbf{M}$ be the $2m \times 2m$ matrix given in the following partitioned form:

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{B} & \mathbf{B} \end{array} \right),$$

where $\mathbf{0}$ is an $(m-1) \times m$ zero matrix. Then $\mathrm{rank}(\mathbf{M}) \leq 2m - 1$, so $\det(\mathbf{M}) = 0$, and using the Laplace expansion with the columns partitioned into $\{1, \ldots, m\}$ and $\{m+1, \ldots, 2m\}$, we obtain

$$0 = \det(\mathbf{M}) = \sum_{k=0}^{m} (-1)^k s_{\gamma^{(k)}} \cdot s_{\delta^{(k)}},$$

from (8). Together with (17), (16) and (11), this implies the result. □

2.3. **A solution to the KP hierarchy.** As an immediate corollary to the content product solution for the Plücker relations given in Theorem 2.2, we now give a content product solution to the KP hierarchy. This result has been previously obtained using different methods by Orlov and Shcherbin [OS] (see also Orlov [Or], equation (1.19)).

Theorem 2.3. *The series*

$$(18) \qquad \Phi := \log \sum_\lambda \left(\prod_{w \in \lambda} y_{c(w)} \right) s_\lambda(q_1, q_2, \ldots) s_\lambda(p_1, p_2, \ldots).$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \ldots$).

Proof. The result is immediate from Theorem 1.3 and Theorem 2.2. □

3. TRANSITIVE ORDERED FACTORIZATIONS AND THE MAIN RESULT

In this section we consider the following general set of transitive ordered factorizations of permutations. For $a_1, a_2, \ldots \geq 0$ and partitions α and β of $d \geq 1$, let $\mathcal{B}_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ be the set of tuples of permutations $(\sigma, \gamma, \pi_1, \pi_2, \ldots)$ on $\{1, \ldots, d\}$ such that

    C1: $\sigma \in \mathcal{C}_\alpha$, $\gamma \in \mathcal{C}_\beta$, and $d - l(\pi_i) = a_i$ for $i \geq 1$, where $l(\pi_i)$ is the number of cycles in the disjoint cycle decomposition of π_i;
    C2: $\sigma \gamma \pi_1 \pi_2 \cdots = \iota$;
    C3: $\langle \sigma, \gamma, \pi_1, \pi_2, \ldots \rangle$ acts transitively on $\{1, \ldots, d\}$.

Let $b_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ be the number of tuples in $\mathcal{B}_{\alpha,\beta}^{(a_1,a_2,\ldots)}$, and let $\widetilde{b}_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ be the number of these tuples in the case in which condition **C3** is not invoked. Instances of such transitive factorizations appear in the combinatorial literature in many places (see, *e.g.*, [LZ], where they are called *constellations*).

As a corollary of Theorem 2.3, we now prove our main result, that a particular generating series for the numbers $b_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ of transitive ordered factorizations is a solution to the KP hierarchy.

Theorem 3.1. *The series*

(19)
$$B := \sum_{\substack{|\alpha|=|\beta|=d\geq 1, \\ a_1,a_2,\cdots\geq 0}} \frac{1}{d!} b_{\alpha,\beta}^{(a_1,a_2,\dots)} p_\alpha q_\beta \, u_1^{a_1} u_2^{a_2} \cdots$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \dots$).

Proof. From the exponential formula for exponential generating series (see, *e.g.*, [GJ1]; interestingly, Hurwitz seems to have been the first person to write this down clearly), we have

(20)
$$B = \log(\tilde{B})$$

where

$$\tilde{B} := 1 + \sum_{\substack{|\alpha|=|\beta|=d\geq 1, \\ a_1,a_2,\cdots\geq 0}} \frac{1}{d!} \tilde{b}_{\alpha,\beta}^{(a_1,a_2,\dots)} p_\alpha q_\beta \, u_1^{a_1} u_2^{a_2} \dots.$$

But, from Proposition 1.1, (9) and (10) we obtain

$$\tilde{B} = \sum_{\alpha,\beta,\lambda\vdash d\geq 0} \frac{1}{d!^2} \left(\chi_{(1^d)}^\lambda\right)^2 p_\alpha \, q_\beta \, g_\alpha^\lambda \, g_\beta^\lambda \prod_{i\geq 1}\left(\sum_{\mu_i\vdash d} g_{\mu_i}^\lambda \, u_i^{d-l(\mu_i)}\right)$$

$$= \sum_{\lambda\vdash d\geq 0} s_\lambda(q_1, q_2, \dots) \, s_\lambda(p_1, p_2, \dots) \prod_{i\geq 1}\left(\sum_{\mu_i\vdash d} g_{\mu_i}^\lambda \, u_i^{d-l(\mu_i)}\right).$$

It is now immediate from (9) and (12) that

$$\sum_{\mu_i\vdash d} g_{\mu_i}^\lambda \, u_i^{d-l(\mu_i)} = \left.\frac{d! u_i^d}{\chi_{(1^d)}^\lambda} s_\lambda\right|_{p_j=u_i^{-1}, j\geq 1} = u_i^d \prod_{w\in\lambda}\left(u_i^{-1}+c(w)\right) = \prod_{w\in\lambda}(1 + u_i\, c(w)).$$

This, together with (18), (19) and (20), gives

(21)
$$B = \left.\Phi\right|_{y_j=\prod_{i\geq 1}(1+j\,u_i),\, j\in\mathbb{Z}}$$

and the result now follows from Theorem 2.3. Note that

(22)
$$\prod_{i\geq 1}(1 + j\,u_i) = 1 + \sum_{k\geq 1} e_k(u_1, u_2, \dots)\, j^k,$$

where $e_k(u_1, u_2, \dots)$ is the kth *elementary* symmetric function in $u_1, u_2, \dots$. $\qquad\square$

4. BRANCHED COVERS, DOUBLE HURWITZ NUMBERS AND m-HYPERMAP NUMBERS

4.1. **Branched covers.** The transitive ordered factorizations in $\mathcal{B}_{\alpha,\beta}^{(a_1,a_2,\dots)}$ also have geometric significance for, by an encoding due to Hurwitz [H], they correspond to branched covers of $\mathbb{C}\,\mathbb{P}^1$ with fixed branched points, say $0, \infty$, and X_i, $i \geq 1$. We require the branching over 0 and ∞ to be α and β, respectively, and the branching over X_i to have $d - a_i$ cycles in the disjoint cycle decomposition, $i \geq 1$. Thus, in **[C1]**, the permutations σ and γ encode the branching over 0 and ∞, respectively, and π_i encodes the branching over X_i, $i \geq 1$. **[C2]** is a monodromy condition and **[C3]** makes the covers connected. The genus g

of these branched covers follows from the Riemann-Hurwitz formula, which in this case gives

$$a_1 + a_2 + \cdots = r^g_{\alpha,\beta}, \tag{23}$$

where, for partitions α, β, and non-negative integer g,

$$r^g_{\alpha,\beta} = l(\alpha) + l(\beta) + 2g - 2. \tag{24}$$

4.2. **Double Hurwitz numbers.** Double Hurwitz numbers arise in the enumeration of branched covers (see [Ok], [GJV]), where they correspond to transitive ordered factorizations in $\mathcal{B}^{(a_1,a_2,\dots)}_{\alpha,\beta}$, in which the branching over each point X_i is simple (a transposition) for $i = 1, \dots, r^g_{\alpha,\beta}$, and trivial (the identity permutation) for $i > r^g_{\alpha,\beta}$. Thus, rescaled for geometric reasons, as in [GJV], the *double Hurwitz number $H^g_{\alpha,\beta}$* is defined by

$$H^g_{\alpha,\beta} = \frac{1}{d!}|\text{Aut } \alpha||\text{Aut } \beta|b^{(a_1,a_2,\dots)}_{\alpha,\beta}, \qquad \alpha, \beta \vdash d \geq 1, \qquad g \geq 0, \tag{25}$$

where $a_i = 1$ for $i = 1, \dots, r^g_{\alpha,\beta}$, and $a_i = 0$ for $i > r^g_{\alpha,\beta}$, and g is defined by (24). Among the results known for double Hurwitz numbers, there is the beautiful and explicit formula for the case $\beta = (1^d)$ and $g = 0$ (see, *e.g.*, [GJ2])

$$H^0_{\alpha,(1^d)} = d!\, d^{l(\alpha)-3}(d + l(\alpha) - 2)! \prod_{i=1}^{l(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!}. \tag{26}$$

We now prove that a particular generating series for double Hurwitz numbers is a solution to the KP hierarchy, as a corollary of Theorem 3.1. This result was first proved by Okounkov [Ok], using a different method, and then more recently by Orlov [Or], and Kazarian [Ka].

Theorem 4.1. *The double Hurwitz series*

$$H = \sum_{\substack{|\alpha|=|\beta|\geq 1, \\ g\geq 0}} \frac{H^g_{\alpha,\beta}}{|\text{Aut } \alpha||\text{Aut } \beta|} p_\alpha q_\beta \frac{t^{r^g_{\alpha,\beta}}}{r^g_{\alpha,\beta}!} \tag{27}$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \dots$).

Proof. From (19), (25) and (27), we obtain

$$\left[\frac{t^{r^g_{\alpha,\beta}}}{r^g_{\alpha,\beta}!} \right] H = \left[u_1 \cdots u_{r^g_{\alpha,\beta}} \right] B.$$

But, from [GJ1] (Lemma 4.2.5(1), *p.* 233), this implies that

$$H = B\big|_{e_k(u_1,u_2,\dots)=\frac{t^k}{k!},k\geq 1},$$

and the result follows from Theorem 3.1. Note that, in terms of the series Φ, we have

$$H = \Phi\big|_{y_j=e^{jt},j\in\mathbb{Z}}, \tag{28}$$

from (21) and (22). □

4.3. m-**Hypermap numbers.** Let m be a fixed positive integer. Define $c_{\alpha,\beta}^{(g,m)}$ by

$$(29) \qquad c_{\alpha,\beta}^{(g,m)} := \sum b_{\alpha,\beta}^{(a_1,a_2,\ldots)},$$

where the sum is over all $(a_1, a_2, \ldots)$ with $a_i = 0$ for $i > m$, and

$$(30) \qquad a_1 + \cdots + a_m = r_{\alpha,\beta}^g.$$

Thus we are considering genus g branched covers with branching over 0 and ∞ specified by α and β, respectively, and arbitrary branching at m other points $X_1, \ldots, X_m$. For geometric reasons, we scale these numbers in the same way as for double Hurwitz numbers above. Hence we define the m-*hypermap number* $N_{\alpha,\beta}^{(g,m)}$ by

$$(31) \qquad N_{\alpha,\beta}^{(g,m)} := \frac{1}{d!}|\mathrm{Aut}\ \alpha||\mathrm{Aut}\ \beta|c_{\alpha,\beta}^{(g,m)}, \qquad \alpha, \beta \vdash d \geq 1, \qquad g \geq 0.$$

We use the term m-hypermap number because the case $m = 1$ yields *rooted hypermaps*, as discussed in a later section. The case $\beta = (1^d)$ and $g = 0$ has been considered by Bousquet-Mélou and Schaeffer [BMS], where they obtained the beautiful and explicit formula

$$(32) \qquad N_{\alpha,(1^d)}^{(0,m)} = d!\, m \frac{((m-1)d-1)!}{((m-1)d-l(\alpha)+2)!} \prod_{i=1}^{l(\alpha)} \binom{m\alpha_i - 1}{\alpha_i}.$$

As a second corollary of Theorem 3.1, we now prove that a particular generating series for m-hypermap numbers is a solution to the KP hierarchy.

Theorem 4.2. *The m-hypermap series*

$$(33) \qquad N^{(m)} = \sum_{\substack{|\alpha|=|\beta|=d\geq 1, \\ g\geq 0}} \frac{N_{\alpha,\beta}^{(g,m)}}{|\mathrm{Aut}\ \alpha||\mathrm{Aut}\ \beta|} p_\alpha q_\beta t^{r_{\alpha,\beta}^g}.$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \ldots$).

Proof. From (19), (31) and (33), we obtain

$$N^{(m)} = B\big|_{u_1=\cdots=u_m=t, u_j=0, j>m},$$

and the result follows from Theorem 3.1. Note that, in terms of the series Φ, we have

$$(34) \qquad N^{(m)} = \Phi\big|_{y_j=(1+jt)^m, j\in\mathbb{Z}},$$

from (21). $\qquad \square$

4.4. **A direct relationship between Hurwitz numbers and m-hypermap numbers.** One relationship between Hurwitz numbers and m-hypermap numbers arises from inclusion-exclusion, as follows. It is straightforward that the summation on the right hand side of (29), subject to (30), and the further restriction that none of $a_1, \ldots, a_m$ is equal to 0, is given by

$$(35) \qquad \sum_{j\geq 0} (-1)^j c_{\alpha,\beta}^{(g,m-j)}.$$

But this forces $a_i \geq 1$ for all $i = 1, \ldots, m$, and so in the case $m = r_{\alpha,\beta}^g$, we must have $a_i = 1$ for all $i = 1, \ldots, m$. This implies that branching over each such X_i is simple, and so, rescaling as in (25) and (31), we have

$$\sum_{j \geq 0} (-1)^j N_{\alpha,\beta}^{(g, r_{\alpha,\beta}^g - j)} = H_{\alpha,\beta}^g.$$

This inclusion-exclusion argument was given in [BMS] for the case $g = 0$ and $\beta = (1^d)$, and enabled them to obtain (26) from (32).

The following result gives another relationship, perhaps more direct, obtained by comparing the generating series H and $N^{(m)}$ in (27) and (33).

Theorem 4.3. *For $g \geq 0$ and partitions α and β of $d \geq 1$, $N_{\alpha,\beta}^{(g,m)}$ is a polynomial in m of degree $r_{\alpha,\beta}^g$, over $\mathbb{Q}$. Moreover,*

$$\left[\frac{m^{r_{\alpha,\beta}^g}}{r_{\alpha,\beta}^g!} \right] N_{\alpha,\beta}^{(g,m)} = H_{\alpha,\beta}^g.$$

Proof. The result follows immediately by comparing (28) and (34). $\qquad \square$

For example, it is straightforward to check that Theorem 4.3 holds in the case $\beta = (1^d)$ and $g = 0$, using the explicit expressions given for these particular double Hurwitz and m-hypermap numbers in (26) and (32), respectively.

We do not know an elementary direct proof of Theorem 4.3. There is a remarkably rich literature on the geometry associated with Hurwitz numbers (the case $\beta = (1^d)$ of double Hurwitz numbers). The fact that Hurwitz numbers arise in Theorem 4.3 as the leading coefficient of m-hypermap numbers (where we can specialize to $\beta = (1^d)$ in the same way) causes us to speculate that much of the geometry associated with Hurwitz numbers may extend to m-hypermap numbers.

5. HYPERMAPS, MAPS AND TRIANGULATIONS IN ORIENTABLE SURFACES

5.1. **Hypermaps in orientable surfaces.** A connected graph embedded in an orientable surface partitions the surface into regions called *faces*, and for *two-cell* embeddings, which are considered here, the faces are homeomorphic to open discs. If the faces are properly *two-colourable*, so faces of the same colour intersect only at vertices (using colours black and white) the embedded graph is called a *hypermap*, where the black faces are *hyperedges* and the white faces are *hyperfaces*. The *degree* of a vertex is the number of adjacent hyperedges, and the degree of a hyperedge or hyperface is the number of sides of edges encountered when traversing the boundary once. We consider *rooted* hypermaps, in which an arrow is drawn on one edge from a tail vertex to a head vertex, so that, moving around the tail vertex, there is a white face to the counterclockwise side of the root edge. For example, on the left side of Figure 1, marked as "a", we give a rooted hypermap in the sphere, in which there are three faces of each colour (the black faces have shaded interiors).

Let $M_{\alpha,\beta}^g$ denote the number of rooted hypermaps in an orientable surface of genus g, with vertex-degrees specified by the parts of the partition α, and hyperedge (black face)
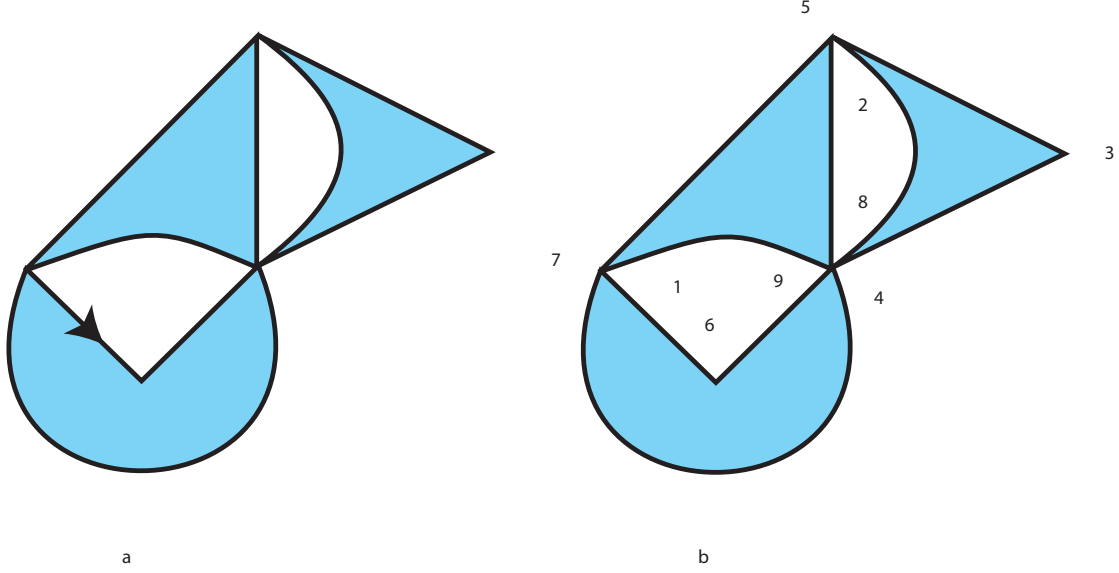
FIGURE 1. A rooted hypermap on the sphere.

degrees specified by the parts of β. It is well-known (see, *e.g.*, [JV1], [JV2], [LZ], [T]) that

$$(36) \qquad\qquad M_{\alpha,\beta}^g = \frac{1}{(d-1)!} c_{\alpha,\beta}^{(g,1)}, \qquad\qquad \alpha,\beta \vdash d \geq 1, g \geq 0.$$

Indeed, this explains the use of the term m-hypermap numbers for $N_{\alpha,\beta}^{(g,m)}$ (which is simply a rescaling of $c_{\alpha,\beta}^{(g,m)}$, *via* (31)). The $(d-1)! : 1$ correspondence implicit in (36) is described as follows: Label the corners of the white faces $1,\ldots,d$ so that 1 is assigned to the corner on the counterclockwise side of the root edge, as encountered when moving around the tail vertex. The remaining $d-1$ labels may be placed arbitrarily, to give the factor $(d-1)!$. From this labelled hypermap, we obtain three permutations in $\mathfrak{S}_d$: σ, γ and π. Each disjoint cycle of σ gives the labels encountered when moving around a vertex in a counterclockwise direction. Each disjoint cycle of π gives the labels encountered when traversing the interior boundary of a hyperface in a counterclockwise direction. Define the label of a black corner (*i.e.*, a corner of a hyperedge) to be the label of the white corner that is encountered in the clockwise direction when moving around their common vertex. Based on this labelling convention, each disjoint cycle of γ gives the labels encountered when traversing the interior boundary of a hyperedge in a counterclockwise direction. Given a rooted hypermap counted by $M_{\alpha,\beta}^g$, it is clear by construction that $\sigma \in \mathcal{C}_\alpha$ and $\gamma \in \mathcal{C}_\beta$, and that $\sigma\gamma\pi = \iota$. Moreover, $\langle \sigma, \gamma, \pi \rangle$ acts transitively on $\{1,\ldots,d\}$, because the underlying graph is connected. This is clearly reversible, and specifies the required $(d-1)! : 1$ correspondence.

For example, on the right side of Figure 1, marked as "b", is one of the labellings of the hypermap given on the left side. For this example, we obtain

$$\sigma = (1\,7)(2\,5)(3)(4\,8\,9)(6), \qquad \gamma = (1\,8\,5)(2\,4\,3)(6\,7\,9), \qquad \pi = (1\,6\,9)(2\,8)(3\,4\,7\,5),$$

and it is easy to verify that, in this case, we have $\sigma\gamma\pi = \iota$.

It is instructive to consider the genus g in (23), which gives the genus for the corresponding branched cover. In terms of the hypermap in the present case (with $m = 1$), condition (23) gives

$$(37) \qquad d - l(\pi) = l(\alpha) + l(\beta) + 2g - 2.$$

Now, in the terminology of Euler's polyhedral formula, the hypermap has $V = l(\alpha)$ vertices, $E = d$ edges, and $F = l(\beta) + l(\pi)$ faces, so a straightforward rearrangement of (37) gives

$$l(\alpha) - d + (l(\beta) + l(\pi)) = 2 - 2g,$$

or $V - E + F = 2 - 2g$, which is Euler's formula. Our use of genus is therefore consistent for covers and hypermaps.

As a specialization of Theorem 4.2, we now prove that a particular generating series for the rooted hypermap numbers $M_{\alpha,\beta}^g$ is a solution to the KP hierarchy.

Theorem 5.1. *The rooted hypermap series*

$$(38) \qquad M := \sum_{\substack{|\alpha|=|\beta|=d\geq 1, \\ g\geq 0}} \frac{M_{\alpha,\beta}^g}{d} p_\alpha q_\beta t^{r_{\alpha,\beta}^g}$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \ldots$).

Proof. From (31), (33) and (36), we have $M = N^{(1)}$, and the result follows immediately, as the specialization of Theorem 4.2 to $m = 1$. $\qquad \square$

Note that the hypermap series M enables us to record vertex degrees and hyperedge degrees separately but not hyperface degrees, so only the total number of hyperfaces (through Euler's formula) is recorded. The analogous series in which hyperface degrees (in addition to vertex and hyperedge degrees) are recorded is *not* a solution to the KP hierarchy.

5.2. **Maps in orientable surfaces.** The specialization from rooted hypermaps to *rooted maps* is by requiring each hyperedge to be of degree 2, and then "collapsing" each hyperedge (black face) to a single edge where, for the hyperedge containing the directed root edge, the collapsed single edge has the same direction as the directed edge. The result is a *rooted map* with the collapsed singled edges as its edges and the hyperfaces as its faces. Let $R_\alpha^{(n,m)}$ denote the number of rooted maps in an orientable surface with n edges, m faces, and vertex degrees specified by the parts of α. Thus α is a partition of $2n$, and the genus g of the surface, by Euler's formula, is given by $l(\alpha) - n + m = 2 - 2g$. Note that, by duality, $R_\alpha^{(n,m)}$ is also equal to the number of rooted maps in an orientable surface with n edges, m vertices, and face degrees specified by the parts of α.

As a specialization of Theorem 5.1, we now prove that a particular generating series for the rooted map numbers $R_\alpha^{(n,m)}$ is a solution to the KP hierarchy.

Theorem 5.2. *The rooted map series*

$$(39) \qquad R := \sum_{n,m\geq 1} \sum_{\alpha \vdash 2n} \frac{R_\alpha^{(n,m)}}{2n} p_\alpha w^m z^n$$

is a solution to the KP hierarchy (in the variables $p_1, p_2, \ldots$).

Proof. Comparing (38) and (39) and applying Euler's formula and (24), we have $\beta = (2^n)$ and $m = 2n - r^g_{\alpha,\beta}$, so

$$R = \left. M \right|_{\left\{ \substack{q_2 = w^2 z, t = w^{-1} \\ q_i = 0, i \neq 2} \right.} , \cdot$$

The result follows immediately from Theorem 5.1. $\square$

For results that correspond to Theorems 5.1 and 5.2 when the generating series are expressed as matrix models see, *e.g.*, [Or] (and for the connection between matrix models and generating series for rooted maps in an orientable surface see, *e.g.*, [J].)

5.3. **Triangulations in an orientable surface of arbitrary genus.** In this final section, we apply Theorem 5.2 to obtain a recurrence equation for rooted *cubic* maps (all vertices have degree 3) in an orientable surface. By duality, these are equivalent to rooted *triangulations*, an important class of maps for the study of surfaces in general.

We begin by defining, for any $\mu \subseteq \{1, 2, 3\}$, the generating series $V_\mu(p_\mu, w, z)$ by

$$(40) \qquad\qquad V_\mu(p_\mu, w, z) := 2z \frac{\partial}{\partial z} R \bigg|_{p_i = 0, i \notin \mu} .$$

In the next result, we obtain combinatorial relationships between various V_μ.

Proposition 5.3.

$$(i) \qquad\qquad V_{\{1,2,3\}}(p_1, p_2, p_3, w, z) = \frac{p_2 w^2 z}{1 - p_2 z} + \frac{1}{1 - p_2 z} V_{\{1,3\}} \left(p_1, p_3, w, \frac{z}{1 - p_2 z} \right),$$

$$(ii) \qquad V_{\{1,3\}}(p_1, p_3, w, z) = wz^{-1} T^2 + \frac{4 p_1 p_3 w^2 z^2}{1 - 4 p_1 p_3 z^2} + \frac{1}{1 - 4 p_1 p_3 z^2} V_{\{3\}} \left(p_3, w, \frac{z}{\sqrt{1 - 4 p_1 p_3 z^2}} \right),$$

where T is the formal power series solution to

$$(41) \qquad\qquad T = z(p_1 + p_3 T^2).$$

Proof. Comparing (40) and (39), we obtain

$$V_\mu = \sum_{n,m \geq 1} \sum_{\alpha \vdash n} R^{(n,m)}_\alpha p_\alpha w^m z^n,$$

with the restriction in the sum over α that all parts of α are contained in μ. Thus, V_μ is the ordinary generating series for the set $\mathcal{V}$ of rooted maps in an orientable surface in which all vertex degrees are contained in μ.

For Part (i):   There are two cases for the maps in $\mathcal{V}_{\{1,2,3\}}$:

    Case 1: All vertices are of degree 2;
    Case 2: Some vertex has degree 1 or 3.

In Case 1, there is exactly one map with k edges for each $k \geq 1$, namely the k-cycle embedded in the sphere. This accounts for the first term on the right hand side of Part (i) of the result.

In Case 2, each such map in $\mathcal{V}_{\{1,2,3\}}$ can be uniquely created by subdividing the edges of maps in $\mathcal{V}_{\{1,3\}}$, replacing them by paths in which all internal vertices have degree 2.

The number of faces is unchanged in this construction. This accounts for the second term on the right hand side of Part (i) of the result, where the external factor $(1 - p_2 z)^{-1}$ is an adjustment for the root edge.

For Part (ii):   There are three cases for the maps in $\mathcal{V}_{\{1,3\}}$:

    Case 1: The map has 0 cycles;
    Case 2: The map has 1 cycle;
    Case 3: The map has at least 2 cycles.

In Case 1, these maps are rooted trees in the sphere, in which all vertices have degree 1 or 3 (each of these has 1 face). If we "*cut*" the root edge, then such trees decompose into an ordered pair of trees from $\mathcal{T}$, which consists of rooted ordered trees on at least one vertex, in which every vertex has up-degree 0 or 2. Let $T^n_{i_1,i_3}$ denote the number of trees in $\mathcal{T}$ with i_1 vertices of up-degree 0, i_3 vertices of up-degree 2, and n edges (so $n = i_1 + i_3 - 1$), and let

$$T := \sum_{\substack{n,i_1,i_3 \geq 0 \\ n=i_1+i_3-1}} T^n_{i_1,i_3} p_1^{i_1} p_3^{i_3} z^{n+1}.$$

Then, T clearly satisfies the functional equation (41), and so we obtain the first term on the right hand side of Part (ii) of the result.

In Case 2, these are embedded in the sphere, with 2 faces, and each vertex on the cycle has degree 3. The edge incident with this vertex that does not lie on the cycle is either inside or outside the cycle, and is also incident with the root vertex of a tree in $\mathcal{T}$. Thus the contribution to $V_{\{1,3\}}$ in this case is

$$(42) \qquad w^2 \frac{2p_3 z T}{1 - 2p_3 z T} + w^2 \frac{2p_3 z \frac{z\partial}{\partial z} T}{1 - 2p_3 z T},$$

where the first term in (42) is for maps with root edge on the cycle (in a canonical, say clockwise direction), and the second term in (42) is for maps with root edge off the cycle. For this second term, we place the root edge on some canonical side of the cycle, say outside, and then direct it in either of the 2 possible directions. But, applying $\frac{z\partial}{\partial z}$ to (41), we obtain

$$(43) \qquad \frac{z\partial}{\partial z} T = \frac{T}{1 - 2p_3 z T},$$

and solving (41) as a quadratic equation in T, we obtain

$$T = \frac{1 - \sqrt{1 - 4p_1 p_3 z^2}}{2 p_3 z},$$

where we have rejected the other root since it is not a formal power series. But this explicit expression for T gives

$$(44) \qquad \frac{1}{1 - 2z p_3 T} = \frac{1}{\sqrt{1 - 4p_1 p_3 z^2}}.$$

Simplifying (42) by means of (43) and (44), we obtain the second term on the right hand side of Part (ii) of the result.

In Case 3, each such map in $\mathcal{V}_{\{1,3\}}$ can be uniquely created by subdividing the edges of maps in $\mathcal{V}_{\{3\}}$, replacing them by paths in which each internal vertex has degree 3. The edge incident with this vertex that does not lie on the path is in the face on either side of the path, and is also incident with the root vertex of a tree in $\mathcal{T}$. The number of faces is unchanged in this construction. The contribution to $V_{\{1,3\}}$ in this case is

$$\left(\frac{1}{1 - 2zp_3T} + \frac{2p_3 z \frac{z\partial}{\partial z}T}{1 - 2zp_3T} \right) V_{\{3\}} \left(p_3, w, \frac{z}{1 - 2zp_3T} \right),$$

where the external factor is an adjustment for the root edge. Simplifying by means of (43) and (44), we obtain the third term on the right hand side of Part (ii) of the result. $\qquad\square$

Now let $\mathcal{S} = \{(n, g) \in \mathbb{Z} \times \mathbb{Z} : n \geq -1, 0 \leq g \leq \frac{n+1}{2}\}$, and define $f(n, g)$ by the quadratic recurrence equation

(45) $\qquad f(n, g) := \dfrac{4(3n + 2)}{n + 1} \left(n(3n - 2)f(n - 2, g - 1) + \sum f(i, h)f(j, k) \right),$

for $(n, g) \in \mathcal{S} \setminus \{(-1, 0), (0, 0)\}$, where the summation is over $(i, h) \in \mathcal{S}, (j, k) \in \mathcal{S}$ with $i + j = n - 2$ and $h + k = g$, subject to the initial conditions

$$f(-1, 0) = \frac{1}{2}, \qquad\qquad f(n, g) = 0, \qquad (n, g) \notin \mathcal{S}.$$

In the next result, we show that the solution to this quadratic recurrence, when rescaled in a simple way, gives the number of rooted triangulations with given genus and number of faces.

Theorem 5.4. *The number of rooted triangulations of genus g, with $2n$ faces, is given by*

$$F(n, g) = \frac{1}{3n + 2} f(n, g), \qquad\qquad (n, g) \in \mathcal{S} \setminus \{(-1, 0), (0, 0)\}.$$

Proof. Let $F(n, g)$ be the number of rooted triangulations of genus g, with $2n$ faces, for $n \geq 1$, $g \geq 0$. These have $3n$ edges, since the sum of the face degrees is twice the number of edges (which also explains why triangulations must have an even number of faces). They also have $n + 2 - 2g$ vertices, which follows from Euler's formula. Thus

(46) $$V_{\{3\}} = \sum_{\substack{n \geq 1 \\ 0 \leq g \leq (n+1)/2}} F(n, g) p_3^{2n} w^{n+2-2g} z^{3n},$$

where duality has been used to interchange the numbers of vertices and faces. Now let

$$\Psi(p_1, p_2, p_3, w, z) := \int V_{\{1,2,3\}}(p_1, p_2, p_3, w, z) \frac{dz}{2z}.$$

Then from Theorem 5.2 and (40), we know that Ψ satisfies (1), in which all partials are with respect to p_1, p_2, p_3 (so setting $p_i = 0$ for $i \geq 4$ is permissible even before differentiation). Let θ denote the substitution operator $p_1 \mapsto 0, p_2 \mapsto 0$. Now apply θ to (1), to obtain

(47) $$\theta F_{2,2} - \theta F_{3,1} + \frac{1}{12}\theta F_{1^4} + \frac{1}{2} \left(\theta F_{1,1} \right)^2 = 0,$$

where, from Proposition 5.3 and (46), the corresponding terms for the solution Ψ are given by

$$\theta\Psi_{2,2} = \frac{1}{2}w^2z^2 + \sum_{\substack{n\geq 1 \\ 0\leq g\leq (n+1)/2}} \frac{3n+1}{2}F(n,g)p_3^{2n}w^{n+2-2g}z^{3n+2},$$

$$\theta\Psi_{3,1} = w^2z^2 + \sum_{\substack{n\geq 1 \\ 0\leq g\leq (n+1)/2}} (2n+1)F(n,g)p_3^{2n}w^{n+2-2g}z^{3n+2},$$

$$\theta\Psi_{1,1} = wz + 4p_3^2w^2z^4 + \sum_{\substack{n\geq 1 \\ 0\leq g\leq (n+1)/2}} 2(3n+2)F(n,g)p_3^{2n+2}w^{n+2-2g}z^{3n+4},$$

$$\theta\Psi_{1^4} = 12p_3^2wz^5 + 384p_3^4w^2z^8 + \sum_{\substack{n\geq 1 \\ 0\leq g\leq (n+1)/2}} 8(3n+2)(3n+4)(3n+6)F(n,g)p_3^{2n+4}w^{n+2-2g}z^{3n+8}.$$

Many of the initial terms may be absorbed into the summations above by using the set $\mathcal{S}$. For example,

$$\theta\Psi_{1^4} = \sum_{(n,g)\in\mathcal{S}} 8(3n+2)(3n+4)(3n+6)F(n,g)p_3^{2n+4}w^{n+2-2g}z^{3n+8}.$$

The result follows by substituting the summation expressions for the four partials into (47), equating the coefficients of $p_3^{2n}w^{n+2-2g}z^{3n+2}$ on both sides of the resulting equation, and rescaling to $(3n+2)F(n,g) = f(n,g)$. $\qquad\square$

For example, applying (45) recursively, Theorem 5.4 gives $F(0,0) = 1$, and

$$F(1,0) = 4, \qquad F(1,1) = 1, \qquad F(2,0) = 32, \qquad F(2,1) = 28,$$

which are consistent with the tables given in [JV3]. Jason Gao (private communication) has checked that the recurrence correctly gives the first 40 terms in genus 0 and 1. The recurrence (45) for triangulations (scaled by $3n+2$ as in Theorem 5.4), appears substantially simpler that the one that has appeared in [G], but we do not not know of a direct combinatorial argument for this recurrence.

Bender, Gao and Richmond [BGR] have been able to use this new recurrence to obtain the explicit asymptotics for triangulations with fixed n and g. Previously, the most explicit asymptotic form for triangulations had been given by Gao [G], which involved a scalar t_g depending only on the genus g. The asymptotics for many classes of rooted and unrooted maps in an orientable surface of genus g was also known up to the same scalar t_g. Thus the fact that the explicit form in [BGR] determines t_g explicitly means that the asymptotics for all these classes of maps are now known explicitly.

References

[BGR]   E.A.Bender, Z. Gao and L.B.Richmond, *Calculating the asymptotic coefficients t_g using a new recursion for rooted cubic maps*, preprint 2008.

[BMS]   M. Bousquet-Mélou and G. Schaeffer, *Enumeration of planar constellations*, Adv. Appl. Math. 24 (2000), 337 – 368.

[G]   Zhi-Cheng Gao, *The number of rooted triangular maps on a surface*, J. Comb. Theory (B) 52 (1991), 236 – 249.

[GJ1]   I.P. Goulden and D.M. Jackson, Combinatorial Enumeration, John Wiley and Sons, New York, 1983 (reprinted by Dover, 2004).

[GJ2]   I.P. Goulden and D.M. Jackson, *Transitive factorizations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. 125 (1997), 51 – 60.

[GS]   I.P. Goulden and Luis. G. Serrano, *A Simple Recurrence for Covers of the Sphere With Branch Points of Arbitrary Ramification*, Annals Comb. 10 (2006), 431 – 441.

[GJV]   I.P. Goulden, D.M. Jackson and R. Vakil, *Towards the geometry of double Hurwitz numbers*, Advances Math. 198 (2005), 43 –92.

[H]   A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 39 (1891), 1 – 60.

[J]   D.M. Jackson, *On an integral representation for the genus series for 2-cell embeddings*, Trans. Amer. Math. Soc. 344 (1994), 755 – 772.

[JV1]   D.M. Jackson, T. Visentin, *A character theoretic approach to embeddings of rooted maps in an orientable surface of given genus*, Trans. Amer. Math. Soc. 322 (1990), 343 – 363.

[JV2]   D.M. Jackson, T. Visentin, *Character theory and rooted maps in an orientable surface of given genus: Face-colored maps*, Trans. Amer. Math. Soc. 322 (1990), 365 – 376.

[JV3]   D.M. Jackson and T.I. Visentin, An atlas of the smaller maps in orientable and non-orientable surfaces, CRC Press, 2001.

[Ka]   M. Kazarian, *KP hierarchy for Hodge integrals*, preprint 2007.

[KL]   M. Kazarian and S. Lando, *An algebro-geometric proof of Witten's conjecture*, preprint 2005, math.AG/0601760.

[Ko]   M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), 1 – 23.

[LZ]   S.K. Lando and A.K. Zvonkin, Graphs on Surfaces and Their Applications, Encyclopaedia of Math. Sci., vol 141, Springer-Verlag, Berlin, 2004.

[MJD]   T. Miwa, M. Jimbo and E. Date, Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras, Cambridge University Press, Cambridge, 2000.

[Ok]   A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Letters 7 (2000), 447 –453.

[Or]   A. Orlov, *Hypergeometric functions as infinite-soliton Tau functions*, Theoretical Math. Phys., 146 (2006), 183 –206.

[OS]   A. Orlov and D.M. Shcherbin, *Hypergeometric solutions of soliton equations*, Theoretical Math. Phys., 128 (2001), 906 – 926.

[P]   R. Pandharipande, *The Toda equations and the Gromov-Witten theory of the Riemann sphere*, Lett. Math. Phys. 53 (2000), 59 – 74.

[S]   R.P. Stanley, Enumerative Combinatorics, Volume 2, Cambridge University Press, 1999.

[T]   W.T. Tutte, Graph Theory, Encyclopedia of Math. and Applns. 21, Addison-Wesley, London, 1984.

[W]   E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Survey Diff. Geom. 1 (1991), 243 – 310.